ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Historia Mathematica 33 (2006) 184–209

HISTORIA MATHEMATICA

www.elsevier.com/locate/yhmat

Algebra in Roth, Faulhaber, and Descartes ☆

Kenneth Manders [1]

Department of Philosophy, University of Pittsburgh, Pittsburgh, PA 15260, USA

Available online 9 November 2005

Abstract

Descartes' "multiplicative" theory of equations in the *Géométrie* (1637) systematically treats equations as polynomials set equal to zero, bringing out relations between equations, roots, and polynomial factors. We here consider this theory as a response to Peter Roth's suggestions in *Arithmetica Philosophica* (1608), notably in his "seventh-degree" problem set. These specimens of arithmetic-masterly problem design develop skills with multiplicative and other degree-independent techniques. The challenges were fine-tuned by introducing errors disguised as printing errors. During Descartes' visit to Germany in 1619–1622, he probably worked with Johann Faulhaber (1580–1635) on these problems; they are discussed in Faulhaber's *Miracula Arithmetica* (1622), which also looks forward to fuller publication, probably by Descartes.

Zusammenfassung

In Descartes' "multiplikativer" Theorie der Gleichungen, dargestellt in *Géométrie* (1637), werden Gleichungen systematisch als Polynome, die gleich Null gesetzt sind, behandelt. Dieses Vorgehen erleichtert das Aufdecken von Beziehungen zwischen Gleichungen, Wurzeln und Polynomfaktoren. Wir betrachten hier diese Theorie als eine Antwort auf Peter Roths Vorschläge in *Arithmetica Philosophica* (1608), insbesondere in seinen Aufgaben zu den Gleichungen siebten Grades. Diese rechenmeisterliche Exempel konzentrieren sich auf multiplikative und andere vom Grade unabhängigen Techniken. Die Herausforderungen werden dadurch raffiniert, daß Fehler eingeführt werden, die wie Druckfehler aussehen. Wahrscheinlich arbeitete Descartes auf seiner Deutschlandreise in 1619–1622 mit Johann Faulhaber (1580–1635) an diesen Problemen. Sie werden in Faulhaber's *Miracula Arithmetica* (1622) diskutiert, wo auch eine weitere Veröffentlichung, wahrscheinlich von Descartes, in Aussicht gestellt wird.

☆ Expanded text of a presentation to the colloquium "Transmission and Transformation of Mathematical Thought," Wuhan, China, October 1998. Revised through January 2005.

E-mail address: mandersk@pitt.edu.

[1] I thank Karine Chemla for repeated encouragement, and Henk Bos, Craig Fraser, and an anonymous referee for detailed comments.

0315-0860/$ – see front matter
doi:10.1016/j.hm.2005.05.003

MSC: 01A45

Keywords: Theory of equations; Cossic algebra; Peter Roth; Johann Faulhaber; Descartes

For its treatment of cubic (and quartic) equations, Cardano's *Ars Magna* is a landmark in the history of algebra. It was eventually to have a significant impact on the German-language arithmetic masters (Rechenmeister). The most influential response in Germany came, via a set of mostly cubic problems in Johann Faulhaber's *Arithmetisch Cubiccossischer Lustgarten* [1604], in Peter Roth's *Arithmetica Philosophica* [1608]; see Schneider [1993, chs. 2–3]. Roth's brief account of Cardano's rules for cubics, his complete solution of all problems in *Arithmetisch Cubiccossischer Lustgarten*, and four sets of new problems (respectively involving quartic, quintic, sextic, and seventh-degree equations) set for some decades the standard in Germany for what could be done in arithmetic and algebra. Roth's impact was ultimately quite spectacular: his book appears to have stimulated Descartes' account of algebra and theory of equations in the *Géométrie* [1637], which later played a key role in the mathematical education of Huygens, Newton, and Leibniz.

Descartes, then 23 years old, traveled from Breda (Netherlands) to southern Germany in mid-1619. As his notebook and correspondence show, in Breda Descartes already had an interest in solving cubic equations. His letter of March 26, 1619 to Beeckman [AT X 154–160] shows Descartes intellectually bold, and proud of his progress on cubics; but his notes show him unaware that his transformations of cubic equations fail to preserve roots [AT X 236–237, 244]. Descartes' algebraic solution methods from the spring of 1619 work only in quite trivial cases. How could this be?

Descartes had learned mathematics from the works of Christopher Clavius. Clavius' *Algebra* [1608] was inspired by Michael Stifel's *Arithmetica Integra* [1543]; both are in the German cossist tradition, but Clavius did not treat cubic or higher-degree equations. Evidently, Descartes had not since encountered any more advanced algebra that would survey how the number of roots of cubics depended on their coefficients or indicate the role of the discriminant—Cardano or Viète, for example, would have made a decisive difference here. He was also at a disadvantage in checking his results. His proposed algebraic transformations led to mostly geometrical solution methods; because these did not give exact (numerical or algebraic) solutions, Descartes was in a poor position to see his approach fail.[2]

Those more up to date in algebra would not have had that problem. Descartes even got the number of (real) roots wrong. Shown the notebook, careful readers of Cardano or especially of Roth's concise account in *Arithmetica Philosophica*, Part I, which was quite explicit about numbers of roots, could tell that Descartes' transformations merged cases with different numbers of roots. With a bit more attention, they would have noticed him simply disregarding denominators in the middle terms of a cubic. We know that Descartes did show this part of his notebook in Germany.[3] That may be why some arithmetic

[2] In fairness to Descartes, these are notes he made for his own use, and in 22-year-old enthusiasm, as the letter shows. Our expectations, in contrast, are formed by works written for paying customers and publication. Who has not made errors in his or her notes? Descartes stands convicted of haste, sloppiness, and lack of advanced education, but not yet of inability to do algebra right.

[3] In Faulhaber's *Continuatio... Kunstspiegels* [December, 1620], Johannes Remmelin attested to Faulhaber's knowledge of a collection of four mathematical instrument designs (text in Manders [1995]). Precisely these designs surrounded these pages

master with a decent collection of geometrical instruments referred him to Roth's book [AT X 241–242]. Descartes evidently studied Roth's book: though notoriously stingy with praise for others' work, he later described it (or Golius quoted him) to Frans van Schooten Jr. as "elegans."[4]

In regards Descartes' relationship to Faulhaber, Lipstorp's (rather rosy and on external historical matters inaccurate) account of Descartes' visit with Faulhaber in [1653, 78–80; AT X 252–253; German transl., with critical discussion: Schneider, 1993, 176–177] was likely Descartes' version via Frans van Schooten Jr.[5] The basic outline of Descartes' relationship to Faulhaber given by Lipstorp is confirmed by various independent evidence on Faulhaber's side, in his *Miracula Arithmetica* [1622] and elsewhere. For detailed discussion of evidence on Descartes's visit to Faulhaber, see Schneider [1993, ch. 9] (who regards it as unlikely), Hawlitschek [1995, sect. 1.2.4], and Manders [1995]. The evidence, taken together, suggests (*pace* Schneider) that Descartes was in Ulm and in extended contact with Faulhaber, probably from June to September 1620. According to Lipstorp, Descartes drew Faulhaber's attention by boasting qualifications in geometrical analysis, and then by promptly solving problems from Faulhaber's *Arithmetisch Cubiccossischer Lustgarten* [1604] that were set him, "moreover adding Rules and general Theorems to serve for solving them and others of the same type." Faulhaber then proposed that they collaborate on Roth's problems.[6]

Faulhaber's only published comment on Roth's problems and his only expository writing on the theory of equations occur in the middle sections of *Miracula Arithmetica*, written by late 1621. These plainly respond to Roth, and display a multiplicative point of view on polynomials and equations not detectable elsewhere in Faulhaber's work. As Schneider [1993, ch. 4] shows, they cover the same material as the theory of equations in the *Géométrie*. They give no hint, however, of Descartes' application to equations of cubic and quartic geometrical problems. In the middle of the multiplicatively oriented part, Faulhaber defers to an expected publication by a young author who, from the details of his description there, can hardly be anyone but Descartes.[7]

---

of Descartes' notebook [AT X 232–241]. The designs are not so remarkable, nor unprecedented, but attribution of just this particular grouping to Faulhaber at this time would be hard to explain except via acquaintance with Descartes' notebook.

[4] See AT X 638, from GroUL Hs 108, folio 20. For this manuscript of Frans van Schooten Jr., including his notes on Golius' mathematical lectures, see van Maanen [1987, pp. 21–24].

[5] Frans van Schooten Jr. was associated with the preparation of Lipstorp's book [Frans van Schooten Jr. to Christiaan Huygens, March 31 1651, HO I 139; Baillet, 1691, Vol. I xiii; AT X 50], and he may have had the story from Descartes himself, or via his father or Golius.

[6] Schneider [1993, 176] doubts Lipstorp's suggestion that Faulhaber took an interest in solving Roth's problems at this time, because the customary time period in which to meet Roth's challenges was long past and Roth was deceased. Faulhaber's response to Roth's death, however, was precisely an attempt to surreptitiously acquire some of Roth's own solutions from his widow (Faulhaber to Sebastian Kurz, December 5 (as) 1617, transcribed in Hawlitschek [1995, p. 297]). Faulhaber's interest is further confirmed by his discussion of the problems in *Miracula Arithmetica*, and especially its urgent tone. Moreover, he stated (p. 52): "Anyone with time enough may now solve all questions posed by the oft-mentioned Roth in his quartic Coss, and decompose them in this fashion; as I indeed gave this challenge to one of my dear friends, who in the mean while has had the appetite to let [solutions to] Peter Roth's examples see the light of day." While the "dear friend" referred to here may well not be Descartes, the passage directly confirms that in this period, Faulhaber asked a student to publish solutions to Roth's problems.

[7] The passage, p. 59, is reproduced in facsimile in Hawlitschek [1995, p. 69], quoted by Schneider [1993, p. 98–99], translated in Manders [1995]. Again, Schneider is skeptical, but has not taken into account the evidence of Faulhaber's contact with Descartes mentioned in footnote 3 above, nor the obvious reasons that Descartes, still to be a candidate for administrative positions in France, would have for insisting on pseudonyms to avoid public association with Rosecrucianism or the Protestant religious fanatic Faulhaber.

Rather than try to assign intellectual credit between Faulhaber and Descartes in detail, I here treat this part of *Miracula Arithmetica* as a time capsule containing results of their extended interaction working on Roth's book.

1. Additive and multiplicative conceptions of polynomials and equations

The most obvious features of polynomials ("cossic quantities") and equations is that they are *sums* or aggregates of terms, and have *roots*, which are typically sought. Sixteenth-century cossists reduced many practical and speculative problems to the problem of finding roots of given polynomials, occasionally ones of quite high degree.

The period around 1600 saw considerable advances in the algebraic theory of equations. We recognize two fundamental mathematical components in these advances. One is the general use of powers of multiple simultaneous unknowns. This became standard in geometrical analysis, but was not straightforwardly available in cossic algebra, as we will discuss in Section 2. Viète elaborates algebra to match geometrical analysis in this respect. The second, our present concern, supplements a traditional "additive" conception of equations by what we call a "multiplicative" conception of equations, a number of considerations based on treating polynomials (in one unknown) as products of ones of lower degree. We start with an analytic summary of these two groups of ideas about equations, using modern notation.

1.1. Additive conceptions of equations

From time immemorable, polynomials have been understood, described, notated, and treated as sums or aggregates of terms. Many relationships, procedures, and properties of polynomials and equations may be understood from this point of view, including the "structural" insights of Viète.

- Relations between notations for equations:
  "setting equal to zero" (Roth, fol. 190; *Miracula Arithmetica* chs. 41–43; *Géométrie*)

  $$x^4 - 4x^3 - 19x^2 + 106x - 120 = 0,$$

  "all signs +" (Cardano, Roth)

  $$x^4 + 106x = 4x^3 + 19x^2 + 120,$$

  "highest equals lower terms" (Stifel, Clavius, Descartes 1619; *Miracula* chs. 36–41)

  $$x^4 = 4x^3 + 19x^2 - 106x + 120,$$

  "variable terms equal constant term" (Faulhaber, Harriot)

  $$x^4 - 4x^3 - 19x^2 + 106x = 120.$$

- The effect of substitutions on coefficients and roots, notably:
  (a) Substituting cx for x (e.g., $-x$ for x to reverse sign of roots).

   (b) Substituting $x + a$ for x, shifting all roots by $-a$, e.g., to eliminate x^{n-1} term.
   (c) Substituting x for x^r (when only powers of x^r occur).

   These substitutions can be carried out whether or not the roots are known.
- The effects, on coefficients and roots, of elimination of terms and unknowns among equations.
- Completing squares:
   (a) Solution of quadratics, conjugacy of their irrational roots.
   (b) Ferrari's reduction of solving a quartic to solving an auxiliary cubic (in Cardano).
- Sign unification: unified treatment of equations differing in $+$ and $-$ signs only; solvability only depends on discriminant conditions.

From an additive (sum-of-terms) point of view, however, the connection between higher-degree polynomials and their roots is obscure; especially when equations are written with nontrivial terms on both sides.

1.2. Multiplicative conceptions of equations

Around 1600, several authors, notably Harriot and Roth, sometimes consider polynomials to result from *multiplying* simpler (i.e., lower degree) ones. This approach provides opportunities to recognize relations between the coefficients of the factors and those of the product polynomial; and hence between the roots of the factors and the coefficients of the product polynomial:

- a is a *root* of $P(x) \iff (x - a)$ is a *factor* of $P(x)$.
- Division of $P(x)$ by $(x - a)$ reduces degree by one; degree limits number of roots.
- Polynomials regarded as products of factors, relating roots to coefficients:
   (a) Root, or constant term of factor, divides constant term of $P(x)$.
   (b) Coefficient of one-but-highest degree term is sum of the roots (and sum of one-but-highest degree terms of any factorization of $P(x)$); absent x^{n-1} term indicates that the sum of roots is zero.
   (c) Sign-reversal of roots may (besides additively) be understood especially concretely from treatment as a product of linear factors.
- Reduce complexity of polynomial by factorization; e.g., Faulhaber and Descartes factor quartic into two quadratics via auxiliary (bi-)cubic.
- Preferred representations of polynomials and equations:
   (a) Factored.
   (b) Sum of terms set equal to zero (ready to factor).
- For integral-coefficient polynomials with irrational roots $m + \sqrt{n}$, the root/factor connection extends, explaining why such roots always come in conjugate pairs:

$$m + \sqrt{n} \text{ is a } \textit{root} \text{ of } P(x) \quad \iff \quad x^2 - 2mx + (m^2 - n) \text{ is a } \textit{factor} \text{ of } P(x).$$

- Sign rule on numbers of positive and negative roots of (fully factored) polynomials.
- Proposals to imagine further roots beyond the real ones, for the polynomial remaining after the factors coming from the real roots are divided out.

Instances of these relationships for quadratics were ancient; and instances for cubics were seen by 1600. Multiplicative orientation toward polynomials (when melded with additive insights), however, helps a unified and more uniform (e.g., *degree-independent*) understanding of algebraic relationships. Degree independence excited Roth, Faulhaber, and Descartes. Multiplicative conceptions provide a non-trivial theoretical perspective on equations, making their nature clear. As we will see, many of these relationships were grasped at least in a partial form by the authors discussed below. In Descartes' hands, they made possible a powerful recovery from a defect of his original idea for classifying geometrical construction problems by the degree of their equations. But on some points, notably in regard to imaginary roots and the (full) fundamental theorem of algebra, these ideas were to receive several centuries of further development.

2. Roth, Faulhaber, and Descartes

2.1. German cossic algebra

Sixteenth-century German algebra provided rules for manipulating and solving primarily linear and quadratic equations, including systems of linear equations, and examples of their application. The influential early text is Christopher Rudolff [1525]; but Stifel's *Arithmetica Integra* [1543] reached a much higher level. The best detailed overview of German cossic algebra is probably still Treutlein [1877]; see also Cantor [1880–1908]. For Roth, see Schneider [1993]; for Faulhaber, see Schneider as well as Hawlitschek [1995].

A crucial limitation of cossic algebra is the lack of notation for higher powers of multiple simultaneous unknowns (rather than a single one). Descartes' geometric and algebraic analysis required overcoming this limitation. Ironically, Stifel [1543] had already proposed quite workable notation for this purpose (see Tropfke [Vol. 3, ch. B3, p. 33 in the 1922 edition] and Cantor [Vol. 2, ch. 62, p. 441 in the second edition]), and Clavius [1608] gave an example (ch. 31, discussed by Sasaki [2003, pp. 79–80]). Roth and Faulhaber, however, do not practice it even where it is needed.

The most striking feature of cossic algebra is that it uses a separate symbol for each power of the unknown; these symbols now strike us as arcane. Given that cossists wanted to be able to consider arbitrarily large powers, a principle was needed to systematically supply symbols and names for higher degrees. The texts considered here use Stifel's "multiplicative" construction. By tradition, the terms *Radix, Zens, Cubus* were used for the first through third powers, each indicated by a specific symbol. Composite powers were formed by juxtaposing symbols, and their names were formed by juxtaposing the corresponding names. For example, the sixth power of the unknown is the *Zensicubus*. Correspondingly, treatment of equations of first degree was referred to as the *Radix* coss, those of the third and fourth degree *Cubic* and *Zensizens* coss, and so on.

This approach required new names and symbols for prime powers; thus there was a special symbol for the fifth power, which was called *Surdesolidus*. From this point on, something more systematic had to be done; Stifel proposed to combine the letters of the alphabet with the usage for the fifth power, so that *Bsurdesolidus* is the seventh power, and the 11th would be *Csurdesolidus*. Stifel's notations are illustrated in Cajori [1928, Vol. 1, pp. 138–146].

2.2. General algebraic claims and suggestions in Roth's Arithmetica Philosophica

In his introduction to the First Book on Cardano's 13 rules for cubics, Roth made striking claims[8]:

- A weak form of the "fundamental theorem of algebra": an equation (in one unknown) has at most as many roots as its degree; some equations have this many roots.[9]

> But as this my ARITHMETICA discusses Cubic, Zensizens, Surdesolid, Zensicubic, and Bsurdesolid cosses, it would have been appropriate to write at somewhat greater length here at the begining about this noble and elegant art: namely, how many a miraculous secret is hidden in the nature of numbers. To indicate just one, about the values [geltungen] of roots in all cosses [equations of all degrees]. For in the Radixcoss (which one otherwise commonly calls Christopher [Rudolff]'s First Rule of the quadratic coss), in no equation can be found more than one value [werth] of the root; but in the equations of the fifth, sixth, seventh, and eighth rule of aforementioned Christopher's, | at most two distinct values [geltungen] of the root can ever be found; in Cubic equations at most three distinct values of the root are available; in the Zensizens coss the most values [geltungen] of the root is 4, in the Surdesolid coss the most values of the root is 5, in the Zensicubic coss the most values of the root is 6, in the Bsurdesolid coss the most values of the root is 7; and so on without end [unendlich] in all subsequent cosses at most as many values are to be found, as with however many the highest quantity in the given cossic equation is designated on the basis of the cossic progression. But here such [assertion] is not to be understood, that therefore every Cubic-cossic equation admits three distinct values, every Zensizens cossic [equation] four, every Surdesolid cossic [one] five; rather all those that admit the most values have that many of them, and moreover not any more.
> (*Arithmetica Philosophica*, fol. 1.recto–1.verso)

- There is a degree-general rule to determine the number of positive and negative roots of a polynomial.

> It should now also be indicated how many always true [waare: positive] and also fictive [gedichte: negative] values of the root there are in any equation. [fol. 1.verso]

- The rule to transform a polynomial to reverse the sign of its roots is degree-general.

> Just so, how any equation, of regardless how many high or low cossic quantities, should be changed and converted [verwandelt und verkehret] in order that in this modified [equation] the true values of the root agree with the fictive ones in the given equation; and conversely, the true values in the given equation agree with the fictive values in the equation into which it is converted. [fol. 1.verso]

Roth's weak form of the fundamental theorem of algebra suggests he had some grasp of a multiplicative conception of equations. In Roth's account of Cardano's rules, however, this is barely confirmed: only Roth's exposition of efficiently finding *integer* roots of cubics might draw attention to the fact that

[8] My translations. Square brackets: an item in English supplements the German text; one in German gives the original of an English term.

[9] This passage was noted by Smith [1929, 292] and Struik [1969, 85]. Roth did not recognize "imaginary" roots here, and he perhaps would not count multiple roots separately, though with a well-developed multiplicative conception one could count linear divisor polynomials instead.

they are factors of the (integer) constant term [1608, fol. 2.recto–2.verso, 4.recto, and 6.recto]. Nor does a multiplicative conception emerge from Roth's Second Book, which consists in a complete treatment of Faulhaber's [1604] problems. Neither is surprising, for among arithmetic masters that would imply that such a conception was implicit in those other works. For Roth's own conception of equations, we must therefore look where he poses his own problems.

2.3. Roth's seventh-degree problems

In his Third Book, Roth gave problem sets involving fourth-, fifth-, sixth-, and last, seventh-degree equations. As customary, the problems were given with answers [Facit]; the challenge to the reader was to show how the answer could be found methodically [reguliert auf zu loesen] and with a rationale. Indeed, by the custom of arithmetic-master challenges, Roth would himself have to have had full and reasoned written solutions; Faulhaber's attempt to acquire some of them confirms this (see footnote 6 above).

The final seventh-degree set was specifically focused on relations between roots and coefficients of equations. In Section 3.1, we translate the problems (I–VI), each followed by an analysis of possible solution methods. With the continuity of our historical story in mind, we restrict ourselves here to recording conclusions. It will be helpful (perhaps indispensable) now and then to look ahead to Section 3.1, to clarify or verify claims that seem puzzling here.

Overall, working the entire problem set required a good grasp of a multiplicative conception of equations. The six equations (after corrections, see below) all factor into linear and quadratic factors with integral coefficients. Typically, there are three integral linear factors/roots and two quadratic ones, at least one with a pair of irrational (real) roots. Roth's equations were therefore obviously constructed by multiplying such factors. (In principle, there was no reason to stop at degree 7, provided one used additional linear factors only, or in some other way avoided having to separate three quadratic or higher-degree factors.) Problems I, II, V, VI involved determining some or all roots of an equation; problem III asked the reader to change the sign of all its roots—from the explicit factorization and its relationship to the roots one can see especially concretely how to proceed. Problem IV asked the reader to determine an equation from seven given roots.

In Problem I, there appears to be no reasoned way to get to the given irrational roots without first removing the three linear factors by polynomial long division, leaving a quartic equation which can then be split into two quadratics. Just finding the three integral roots is not good enough, one needs the explicit quartic equation. Roth presumably expected the reader, following his example, to solve the quartic by Ferrari's (completing-the-square) method from Cardano; but a multiplicative algebraic analysis equating the quartic to a product of two quadratics with undetermined coefficients (see below) is also possible. The key to a multiplicative approach to Problems V and VI is to recognize how partial specification of an irrational root constrains a quadratic factor of the given polynomial (there is also an additively motivated approach). Finally, Problems I, II, III, V, and VI would be most tedious indeed if one did not recognize that integral roots as well as constant terms of quadratic factors must evenly divide the constant term of the equation.

There were, however, errors in some printed polynomials (II, III, IV). Once corrected, they can look like multiple single-digit misprints in large coefficients. Detecting, locating, and correcting errors while solving the problems enhanced their challenge. Was this Roth's intention? After analyzing the impact of the misprints on each problem separately in Section 3.1, we compare them in Section 3.2. The printed coefficients are off by (II) $4998x$ and $4999x^2$; (III) $98x^2$, $49x^2$, $48x$; (IV) $198x$ and $199x^2$. This pattern could not plausibly be coincidental; one must take the corrections as skills intentionally required by the problems. Not only did the "misprints" adjust the problem demands, they self-certified this: the successful solver (or customer paying Roth for solutions) could tell after the fact, from the unlikely pattern, that the misprints were intentional. Roth thus had a built-in defense that his problems were printed with proper care![10]

The plausible route to "correcting" Problem II—see there—takes the five printed roots, and (for each plausible way of making the irrational roots conjugate) compares their sum and product with the sixth-degree term and constant term of the printed septic, assuming these numerically small coefficients are correct. The discrepancies give the coefficients of another quadratic factor, and then all the factors can be multiplied to find the "intended" seventh-degree polynomial; it agrees with the printed information except for pairs of single digits in the largest coefficients.

To correct Problem III, compare its printed sum-of-roots and product-of-roots terms (again, assuming these small-coefficient terms correct) with those in Problem II; this suggests that the roots (and constant terms in factors) in III might be the same as those in II, except for one root doubling with sign reversal reducing their sum by 9. Indeed, changing the root 3 in II to -6 has these effects. One cannot, however, verify these conjectured roots in Problem III by evaluation or division because one still lacks the correct polynomial. Hence, the only route is to reconstruct it by multiplying its supposed factors together. The resulting septic agrees overwhelmingly with the printed ones.

These corrections thus also require multiplicative insights. Overall, Roth's seventh-degree problems are now seen to form a set of six distinct problems of the form: given some coefficients and some roots of a polynomial (which, however, is restricted to products of quadratic and linear factors with integral coefficients), find the remaining ones. Together with Roth's theoretical suggestions in Book I discussed above, correcting and answering these six problems involves some control of the first seven points of multiplicative insight into equations we specified above. Only the eighth, on nonreal roots, are not implied here (despite suggestive remarks in Cardano and Descartes).

In addition, intelligibly exploring and correcting these problems requires quite a bit of facility with *algebraic analysis:* setting out and evaluating algebraic relations involving initially unknown quantities. Below, we discuss the kinds of challenges this could be expected to pose for cossic algebra.

2.4. Theory of equations in Faulhaber's Miracula Arithmetica [1622]

Sections 36–44 of *Miracula Arithmetica*, pp. 43–73, on equations, differ strikingly from anything else published by Faulhaber. One immediately discerns a response to Roth's seventh-degree problems [Schneider, 1993, ch. 3], and the material also prefigured [Schneider, 1993, ch. 4] Descartes' theory of

[10] Published problems served arithmetic masters as advertisements and public challenges, and they had long recognized how important it was to avoid printing errors in them. Cantor [1900, Vol. 2, p. 447] recounts Stifel's insistence in personally supervising printing a problem he appended to his 1553 edition of Rudolff's *Coss*; quoting Stifel: "because in these matters the most insignificant error has an unusually broad impact" [weil hierbei der unbedeutendste Fehler von ungemeiner Tragweite sei].

In *Miracula Arithmetica*, Faulhaber commented that some of these problems were "however, printed very wrong" [zwar zum theil im Text sehr falsch gesetzt worden] (p. 60). Just possibly, the action-suggesting "gesetzt *worden*," rather than just "gesetzt," recognized intentional "error?" In any case, the comment indicated something unusual.

equations in the *Géométrie*. In between two multiplicatively motivated topics, Faulhaber deferred to a visitor (plausibly Descartes; see above in the Introduction) to publish on related matters.

Miracula Arithmetica [p. 46] claimed an important advance in uniformity: "Now Cardano, and after him Peter Roth ... also showed how to solve the Cubic Coss .... But they ... divided up the Cubic Coss into thirteen separate rules. Because of which, the invention of the General Method remained hidden from them." Indeed, these sections avoid, in the fashion we have come to associate with Descartes' algebraic style, separating cases based on the signs of terms. Already in Breda, Descartes expected sign variants not to make a big difference [AT X 155 6–10, 240 1–4].

Chapters 36–39 show how to eliminate the one-but-highest term of equations of arbitrary degree, by substituting $x + a$ for x [my notation, not in the text].[11] In evaluating powers $(x + a)^i$, tables including "Pascal's" triangle were used systematically to keep track of terms and their signs. Zero coefficients marked positions to keep different powers of the unknown in distinct columns. Faulhaber saw eliminating the penultimate term as proper preparation for solving equations of any degree (see further at footnote 15), including the multiplicative approach to factoring quartics for Roth's Problems I, II. An additively motivated approach to Roth's Problems V and VI also uses these substitutions.

Fulfilling Roth's hint [1608, Part I], ch. 40 indicates, for any equation, how to tell how many positive ("ware") and negative ("gedichte") roots it has, "Descartes' rule of signs." These terms for types of roots were Roth's, following Cardano.[12]

Chapter 41 reduces the degree of an equation by dividing by a linear factor if a root is known (needed for Roth's Problems I and II), and obtains an equation from its roots by multiplying binomials (Problem IV). Hawlitschek [1995, pp. 68–69] reproduces *Miracula Arithmetica*, pp. 58–59, illustrating Faulhaber's notation in these sections. Here, Faulhaber pointed out that this technique would easily solve Roth's Problem IV, and commented on Roth's Problem V. Faulhaber now deferred to Descartes' anticipated publication of such (and further) matters. (See footnotes 7 and 18.)

Chapters 42–44, pp. 60–70, show by indeterminate analysis how to factor a quartic equation into two quadratic ones (as needed in solving Roth's Problems I, II) using a solution to an auxiliary cubic. This multiplicatively motivated method in *Miracula Arithmetica* replaces Ferrari's additively motivated completing-the-square approach, which Roth [1608, Part II] used to solve the two quartic problems in Faulhaber [1604], and these same two examples are treated here. Then the auxiliary sextic/cubic is described, verbally but in general, including several sign cases. Chapter 44 illustrates how to compute the sextic/cubic directly from the quartic, omitting the derivation. Schneider [1993, ch. 4.2] discusses this factorization, but we will make some different points.

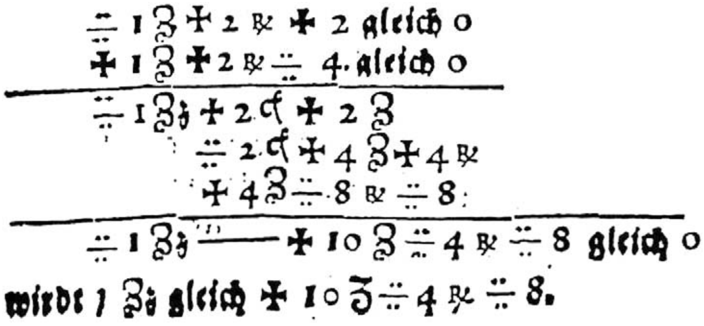
To find its quadratic factors, the quartic, cubic term already removed (ch. 38), is set equal to zero. For example, $-x^4 + 10x^2 - 4x - 8 = 0$ (pp. 60–63).[13] Faulhaber multiplied the factors $-x^2 + 2x + 2 = 0$

[11] This corrects and transcends Descartes' earlier efforts to transform cubics in his notebook.

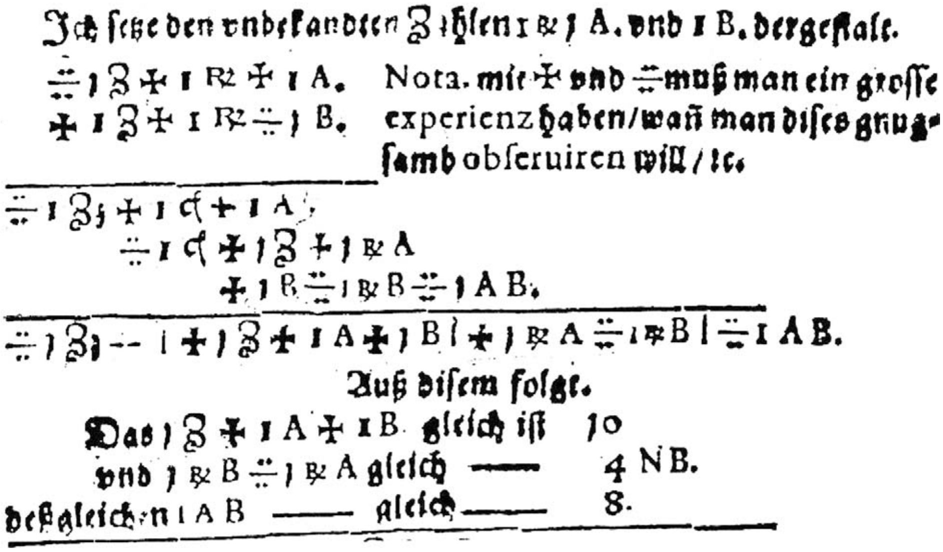
[12] The procedure can overestimate. Faulhaber's 10th degree example, p. 57, has only two roots, not six, as he claims it "must have." Curiously, this could be seen by the methods in *Miracula Arithmetica*: substituting $(x - 1)/2$ for x on p. 55, Faulhaber got a polynomial with only even powers of x, i.e., one of fifth degree in x^2. Replace x^2 by x, so only positive roots now count. He knew one root $441 = 21^2$ a priori, and dividing gives the integral quartic $36555290355 + 82891814x + 187964x^2 + 426x^3 + x^4$. This plainly has no positive root. Hence the original 10th degree polynomial has one "wahre" and one "gedichte" root: $(21 - 1)/2$ and $(-21 - 1)/2$.

[13] We transcribe cossic symbols by exponents of x (but see the comments), omitting coefficients equal to 1, which Faulhaber systematically supplied, e.g., $-1x^2 + 1x + 1A$.

and $x^2 + 2x - 4 = 0$ explicitly, to show the quartic as their product (p. 60):


÷ 1 ℨ + 2 ℞ + 2 gleich 0
+ 1 ℨ + 2 ℞ ÷ 4. gleich 0
÷ 1 ℨℨ + 2 ℭ + 2 ℨ
÷ 2 ℭ + 4 ℨ + 4 ℞
+ 4 ℨ ÷ 8 ℞ ÷ 8.
÷ 1 ℨℨ —— + 10 ℨ ÷ 4 ℞ ÷ 8 gleich 0
wirdt 1 ℨℨ gleich + 10 ℨ ÷ 4 ℞ ÷ 8.

To extract factors from the quartic, he "multiplied" $-x^2 + x + A$ and $x^2 + x - B$ using the same format (p. 61),


Ich setze den vnbekandten Zahlen 1 ℞ 1 A. vnd 1 B. dergestalt.
÷ 1 ℨ + 1 ℞ + 1 A. Nota. mit + vnd ÷ muß man ein grosse
+ 1 ℨ + 1 ℞ ÷ 1 B. experienz haben/wañ man dises genugsamb obseruiren will/ etc.
÷ 1 ℨℨ + 1 ℭ + 1 A
÷ 1 ℭ + 1 ℨ + 1 ℞ A
+ 1 B ÷ 1 ℞ B ÷ 1 A B.
÷ 1 ℨℨ -- | + 1 ℨ + 1 A + 1 B | + 1 ℞ A ÷ 1 ℞ B | ÷ 1 A B.
Auß disem folgt.
Das 1 ℨ + 1 A + 1 B gleich ist 10
vnd 1 ℞ B ÷ 1 ℞ A gleich —— 4 NB.
deßgleichen 1 A B —— gleich —— 8.

but instead of

$$-x^4 + x^2 + x^2A + x^2B + xA - xB - AB,$$

he obtained (!)

$$-x^4[|] - - \mid +x^2 + A + B \mid +xA - xB \mid -AB.$$

Here, the "|" separate columns.[14] He now equated column-by-term with the original quartic polynomial, to get auxiliary equations $x^2 + A + B = 10$, $xB - xA = 4(!)$, and $AB = 8$.

The reader was evidently expected to see, by comparing the two multiplications using the same format, that the unknown in the auxiliary equations plays the role of the (equal) coefficients of linear terms in the quadratic factors. We would use a separate unknown for this role, multiplying

$$\left(-x^2 + yx + A\right)\left(x^2 + yx - B\right) = -x^4 + \left(y^2 + A + B\right)x^2 + (yA - yB)x - AB.$$

The right-hand side plays the role of the column-separated expression above. Comparing term by term with the original quartic would then give $y^2 + A + B = 10$, $yB - yA = 4$, and $AB = 8$. In cossic notation, these conditions were indistinguishable from those in the text. Of course, this reading would still not render the notation consistent in our eyes, as the "x^4-column" entry should then be -1, not $-x^4$.

Faulhaber eliminated A and B (pp. 61–62) to obtain an auxiliary sextic: $x^6 = 20x^4 - 68x^2 + 16$. Rewriting this as a cubic (p. 62 bottom), $x^3 = 20x^2 - 68x + 16$, he noted (p. 63 top) the root $x = 4$ (e.g., try divisors of the constant term 16); hence for the sextic, $x = 2$. This allowed him to solve $A = 2$ and $B = 4$. These steps do not involve significant conflicts between multiple uses of the cossic symbols; we would use a separate variable z to replace y^2, but reusing the cossic symbols in such replacements was routine and did not lead to ambiguity. The values found for x (our y), A, B suffice to give the quadratic factors. No further explanation was given.

In *Miracula Arithmetica* we thus find, in an admittedly awkward form, the indeterminate analysis that justifies the method of factoring quartics given in the *Géométrie*, pp. 383–384. (Perhaps its occurrence here explains why Descartes omits the analysis there, given Descartes' claim that he only wrote out what went beyond what others had published?) Multivariate analysis stretches cossic algebra where it is thinnest: simultaneous use of several unknowns.

Auxiliary unknowns A, B were not especially novel here: a standard type of practical arithmetic problem ("Four merchants together buy a party of cheese ...") involved several linear equations in several unknowns, and it was customary to augment the cossic unknown by $A, B, C, \dots$ for this purpose (e.g., [Petri, 1583–1663]). Schneider [1993, p. 69] documents this usage in Faulhaber [1604] and Roth. In these applications, all unknowns (including the one given by a cossic symbol) occurred linearly and had fully analogous roles.

Miracula Arithmetica here used A and B in analogous roles, and without higher powers. Their role was now, however, very different from that of the cossic unknown of the original quartic, nor were they treated at all parallel to the cossic unknown in the auxiliary sextic and cubic. The products xA, xB, and AB went beyond the usual, but the cossist could notate them.

If this much innovation was possible, then one could also imagine notating the expression $(-x^2 + yx + A)(x^2 + yx - B)$ using some additional letter (C?) for the linear occurrences of the auxiliary unknown, our Cartesian y, though this would violate symmetry between the auxiliary unknowns. It might be best to think of the formulas of *Miracula Arithmetica* in this way; but merely keeping the extra unknown "in mind" at the relevant positions, in order to avoid flaunting disanalogy to the other auxiliaries A and B, and to ease transition to the second group of formulas where the cossic symbols indicated powers of this

[14] The long-multiplication computation is again shown in full for the second example on p. 64. This leaves no doubt that the factor x^2 is deliberately absent from the terms "x^2A" and "x^2B" we would expect in the product, not somehow lost in adding up or regrouping intermediate results.

auxiliary unknown. The cost of not somehow making the auxiliary unknown explicit in evaluating the product was, however, that it took a verbal "road map" in addition to the displayed formulas to make the rationale of the factoring procedure clear.

It is not unusual for the cossic unknown to have a different significance in different polynomials in the same display. In particular, substitutions for the unknown reusing the cossic symbols were common, for example, in Descartes' notes of March 1619 [AT X 234, 236], above in *Miracula Arithmetica*, chs. 36–39, and here in reducing from the sextic with only even powers of the unknown to a cubic. Such changes were usually accompanied by some verbal explanation; here (p. 62): "Here one puts the squares aside again, that is, one takes the square root of the cossic quantities." Faulhaber did not, however, remark on the transitions between the original cossic unknown and the auxiliary one (our y).

The representational challenge that these cossists did not meet directly, in spite of Stifel's notational proposal for doing so mentioned in Section 2.1, is to provide higher powers of two unknowns (our x, y) in the same expression, here in the result of multiplying the two quadratic factors in general form. Hence the column separator expression, in which the powers of the unknown of the original quartic—after all, we still can see them in the displayed quartic—were expressed positionally, by the columns, leaving the cossic powers available for the auxiliary unknown. (Why, indeed, not set up the whole multiplication of quadratics in column separator form from the start?)

Modern algebraic notation has plenty of letters for knowns and unknowns with arbitrary powers, and so needs no special techniques to reuse a single power-capable unknown: one can devote a separate unknown to each separate quantity or role. In addition, transitions between expressions, for example, substitutions for an unknown, may often themselves be made explicit by equations rather than left as verbal comments; this brings such transitions themselves within the scope of algebraic manipulation.

These possibilities are the main conceptual advantages of the modern algebraic notation over the cossic tradition. The effect of their absence may be measured in the lack of rigor in Descartes' March 1619 transformations; in the borderline comprehensibility of analysis of quartic factorization (worse if the cubic term is not first removed); and in the inaccessibility to analysis of possible reductions of sextics (see below). Until degree-general patterns become central, these conceptual advantages are much more important than the benefit of exponent notation for specific powers of a single unknown: a little practice makes cossic notation for specific powers easy to use.

Faulhaber concluded the discussion of factoring in ch. 44 by claiming (incorrectly) that this approach to quartics would extend to higher-degree equations, provided one used more unknowns $x, A, B, C, D, \dots,$ in the derivation, a claim repeated for arbitrary degree in *Academia Algebrae* [1631, fol. E.ii]. Faulhaber did not clarify what factorizations this might give for higher degrees: perhaps sextics into three quadratics using x, A, B, C? perhaps sextics into two cubics? The list of unknowns Faulhaber gives suggests that he is not attending to the need for powers of increasingly many simultaneous nonlinear auxiliary unknowns such as y in the multiplication formula.[15]

[15] In the *Géométrie*, Descartes made a strikingly parallel incorrect claim, that "there is a general rule to reduce to the third degree all fourth-degree problems, *and to the fifth degree all sixth-degree ones*" [1637, p. 323, my italics]. When Dozem asked him about this in 1642, Descartes wrote that he had pretty much forgotten finding some algebraic procedure to split a sextic using a quintic, and (facing Dozem's report that separating a quadratic factor of a quintic already leads to a 10th degree auxiliary) made an incoherent suggestion [AT III 557–560]. Also an unexamined idea from the early 1620s? (Descartes must have had his early algebra notes, item D of the Stockholm inventory [AT X 8], with him in the Netherlands. Presumably, then, these did not explicitly pursue the higher-degree cases either.)

Chapter 44 continued [pp. 70–73] with an example of solving two simultaneous higher-degree equations in two unknowns (x cossic; a by repetition: $aaaa$ for a^4) by polynomial elimination; following the steps of the elimination in detail, however, would try the patience of a stone. The example shows that the elimination technique goes beyond application to factoring.

All results in these chapters were degree-general—or Faulhaber (and Descartes?) took them to be so. All explicitly responded to Roth, if perhaps not always quite as Roth may have intended. Faulhaber remarked that Roth's seventh-degree problems involved factors of polynomials, though printing errors spoiled the factorizations (p. 60). Of the issues raised by Roth, only sign reversal of roots was not spoken to in *Miracula Arithmetica*.

The repetition notation *aaaa* is familiar from Descartes' later writings; the present passage seems to be its earliest occurrence. Its origin is unclear. It is not found elsewhere in Faulhaber's writings except in *Academia Algebrae* in this same connection; nor in the work of a Lienhard Sutor to which Faulhaber referred here [Manders, 1995, footnote 12]; nor in what we have of Descartes' notes from this period. Stifel [1543, fol. 61B] had proposed a repetition notation for powers of distinct (unknown) quantities, but with capital letters $A, B, \ldots$; see Cajori [1928, Vol. 1, p. 144].

Another puzzling point about this notation: if he had a notation for powers of multiple simultaneous unknowns in hand here, one wonders why Faulhaber did not derive the auxiliary conditions for factoring the quartic using this notation (with a for our y), and so explain the method. That would have required only a few extra symbols in the existing multiplication layout of chs. 42–43.

In contrast, Descartes was to make elimination of auxiliary unknowns in polynomials a central element of his geometrical method, in order to obtain an equation of one unknown for (determinate) geometrical construction problems. That he did so already in the early 1620s is made plausible by the following considerations taken together: by his March 1619 proposal to use the degree of such equations as a criterion of constructional complexity of problems, together with his later claims of problem-solving and (probably only preliminary) classificational success in the *poêle*, which would have required polynomial elimination; by his mention of multiple equations in Rule XIX and the elimination step in Rule XXI; and by his early discovery of the pertinent uniform construction of roots of quartic equations by intersecting a circle and a parabola (which only becomes urgent after cubic and quartic equations of geometrical problems have been obtained), announced in Paris in the mid-1620s but mentioned by Lipstorp already in the Faulhaber passage.

2.5. Theory of equations in Descartes' Géométrie

We may refer to Bos [2001, ch. 27] for a detailed presentation of Descartes' highly innovative theory of equations. Here we review in relation to Roth and *Miracula Arithmetica*, as also discussed by Schneider [1993, ch. 4].

The theory of equations in the *Géométrie*[16] resolutely and from the outset took a multiplicative orientation toward polynomials to be fundamental to understanding their "nature." First, one should always set polynomials equal to zero (371). Then, the fundamental theorem of algebra (weak form) is stated. It is motivated by the root/factor connection, emphasizing polynomial long division rather than evaluation as the method for recognizing roots (372–373). Next come the sign rule and the reversal of the signs

[16] Page references to [Descartes, 1637]. In Descartes' *Oeuvres* AT VI, 444–463, in order of mention: AT VI, pp. 444, 444–446, 446, 447–448, 449–452, 452–453, 454–456, 457–461.

of the roots (373); the substitution of $x + a$ for x, its effects, and its applications (374–378), notably to removing the x^{n-1} term; and the substitution of cx for x and its effects and applications (379–380). All this is degree-general. Finally, Descartes considered reducing the degree of a polynomial by finding its factors, first for degree 3 (380–382) and then for degree 4 (383–386) using the multiplicatively motivated method from *Miracula Arithmetica* rather than the additively motivated one found in Cardano. Descartes exploited the fact that the constant term of a factor divides that of a polynomial to limit the search for factors. He gave formulas for the factorization of the general quartic into two quadratics. All this was illustrated for cubics and quartics with arbitrary coefficients. The reduction is almost entirely algorithmic!

The *Géométrie* went beyond Roth and *Miracula Arithmetica* on several points. It introduced setting equations equal to zero as default; and stated the rule of signs more cautiously, allowing that there might be fewer than the indicated number of roots. Descartes extended factoring techniques from polynomials with integral coefficients to ones with integral polynomial coefficients (viz., polynomials in the known quantities of a geometrical problem). He also went flexibly back and forth between different levels of algebraic generality, as when he analyzed factoring for general quartics first and then substituted (still nonnumerical) coefficients arising from specific geometrical problems.

Techniques for deciding when (quartic) equations may be factored, and for doing so, are crucial to Descartes' long-term project since 1619 [AT X 156–157] of classifying the constructional complexity of geometrical construction problems by the degree of their equations, the main program of the *Géométrie* [Bos, 1990]. Remarkably, what Descartes needs and achieves here are criteria for reducibility/irreducibility into factors with coefficients *constructable* by circles and lines from those of the given equation. For he can only match Pappus' ancient construction by circles and lines of a certain naively quartic neusis problem by constructably factoring its equation into quadratics, and the theory of equations is introduced in the *Géométrie* for this specifically. In doing so, he went beyond what the analytically advanced Viètian Ghetaldi could do even by 1630, as discussed by Bos [2001, Section 5.4].

The *Géométrie* put the multiplicative conception of polynomials center stage (together with further elements of uniformity in treatment of signs and degree), after intimations in Roth and key responsive developments in *Miracula Arithmetica*. Nonetheless, it is hard to miss that each topic in Descartes' theory of equations was already announced in Roth or developed in *Miracula Arithmetica*, or both. (Harriot's multiplicative treatment of equations was earlier, apparently from the 1590s, but its delayed publication [Harriot, 1631] precludes a role in forming Descartes' approach.)

2.6. Possible influence of Viète

It has been suggested that Descartes' theory of equations was influenced in fundamental aspects by Viète, or was based on Viète's ideas, or that Descartes directly or indirectly adopted the Viètian approach to algebra. Descartes's own statements on his relationship to Viète's work tend to deny this, and nothing explicitly in the historical record known so far contradicts him. Beyond this, the matter is not easily assessed, notably because the interval between Viète's publications in the 1590s and Descartes' formative period 1619–1620, let alone up to the final composition of the *Géométrie* in 1636, allows myriad indirect influences (including publications by Viète's students such as Ghetaldi) that we cannot ever hope to exhaustively trace. Sasaki recently gave an excellent review [2003, ch. 5.3]; see also indications in Bos [2001, pp. 268, 297, 396]. We comment briefly in light of the present detail on Descartes's formative history through 1620. We can divide into three periods: Descartes in France until 1618, at Breda 1618–1619, and in Germany 1619–1621.

Known evidence of Descartes' early French career, and the rarity of Viète's works even there at the time, aside, Descartes' writings in Breda through March 1619 known from the *Cogitationes Privatae* and the Beeckman materials [all in AT X] indicate detailed and definite limitatations on Viètian influences on Descartes' mathematical thinking in early 1619, either direct or indirect! (This limits the relevance of even future investigations into possible influences on Descartes' algebra in France before the Breda period.)

Indeed, these materials put limits on the intellectual influences on Descartes at the Breda military academy, where his interests turned strongly to scientific and mathematical matters and many sources might in principle have been available to read or from instruction and conversation. Descartes' March 1619 geometric attempts to solve cubics could remind one of Viète; and his ideas on geometrization of physics no doubt reflect some kind of intellectual influences in Breda. In regards *algebra*, however, Descartes' notes, mostly for his own use, show him a cossist pupil of Clavius, bumbling on transforming cubics without anything reminiscent of Viète's notation or ideas (except an attempt to indicate general coefficients in equations that is so primitive that it establishes the lack of influence), or even ideas of Cardano and his intellectual descendents.

This brings us to Germany. Viète's work was generally unpublished or unavailable in Germany in this period. Clavius (German-trained) states in his *Algebra* [1608, again in the 1612 Mainz edition], with which Descartes was familiar, that he has heard of Viète's treatment of solutions of cubics, but has not been able to consult it (not even in Rome?) [Sasaki, 2003, p. 75]. For Roth, Faulhaber, and Descartes in Germany, the closest indications may be from Faulhaber's mathematical bookseller's correspondence with Kurz, which appears to give a reasonably complete record from 1604 to 1633. In September 1608, Faulhaber asked Kurz for any information (works in German or Dutch) about Viète, Bombelli, and Petrus Nonnius [Hawlitschek, 1995, p. 71]. Having traveled the intellectual world after his studies with Faulhaber, Johann Schreck (Terrentius) told Faulhaber in May 1609 of "amazing things of students of Viète's in Rome and Paris." By June, Schreck had told him Viète could solve equations of arbitrary degrees [Hawlitschek, 1995, p. 79]. There was no further mention of Viète in the extant correspondence, although his work would have had high interest for Faulhaber. Neither Roth's nor Faulhaber's works show the influence of Viète. Apparently, no algebraic works of Viète had reached Faulhaber and his associates by the time Descartes was in Germany. Finally, we have another Cartesian mathematical text from the German period, probably after the contact with Faulhaber and inspired by his and Remmelin's interests (see Manders [1995, Section IV] for the context), the *De solidorum elementis* [AT X]; it uses algebra to study geometry, nothing multiplicative, but again does not show Viètian influence in analysis or notation, providing a later limit (1621–1622?).

The only traces of a multiplicative conception of equations I know of in Viète's published work are found in the final section XIV of his *De Aequationum recognitione et emendatione* [Viète, 1615; transl. Viète, 1983, pp. 309–310], discussed by Sasaki [2003, pp. 242–243]. Propositions I–IV there gave the roots of equations of the second to fifth degrees in relation to the equations in the form we would obtain by multiplying linear factors. For example, B, D, G are said to be roots of

$$A^3 - A^2(B + D + G) + A(BD + BG + DG) = BDG \quad \text{(transliteration Witmer).}$$

Viète did not set equations equal to zero, but the somewhat subtle sign choices were all correct. Viète gave no explicit mathematical comment, and formulated none of the further characteristic multiplicative insights. Viète only said, "I have treated [elsewhere] at length and in other respects with the elegant reasoning behind this beautiful observation, [so] this must be the end and crown [of this work]" (translation and interpolations by Witmer). Evidently, Viète had grasped perhaps a good bit of a multiplicative conception and was struck by it.

Unfortunately, this material was not published and I do not know how it might have had an influence; the 1615 publication was too late to have influenced Roth's ideas. Roth [1608] in any case went far beyond Viète's later published discussion: a relatively sophisticated version of the fundamental theorem of algebra, understanding the degree as only an upper bound on the number of (what we would call real) solutions; intimation of the sign rule for the number of positive and negative such solutions; recognition that one would have to allow products of both linear and quadratic base polynomials; commensurate factorization techniques for polynomials; subtle techniques relating roots and coefficients of polynomials in a number of ways. Roth himself indicates the pre-Viète *Ars Magna* of Cardano as his major source, but multiplicative conceptions are not found there.

Finally, to Descartes' own comments. Descartes has perhaps been thought ungenerous to his sources, but Roth's book is mentioned in Descartes' notebook, and Descartes did praise several mathematical sources to his pupils in Holland, including Roth (footnote 4 above). He also mentioned to them his mathematical contact with Faulhaber in connection with Roth's book (footnote 5). Thus, Descartes did speak up about mathematical sources that mattered to him early on, but Viète and possible intermediates are not among them.

Taken together, these considerations seem to me very strong evidence for the following picture. As Mahoney [1980] pointed out, Viète initiated a "structural" study of equations, a study based on algebraic form. While he attained certain elements of a multiplicative conception, this was hardly reflected in his (even posthumously) published work. Nor does anything beyond rumors of Viète's work on equations seem to have reached Descartes' German sources (incredibly, perhaps, not even Clavius). By 1608, Roth published indications of a rather rich multiplicative conception of equations. It was not only independent of Viète's earlier work (as later published), but went significantly beyond it. Probably in 1618–1619, Descartes worked this through in Germany, probably with Faulhaber, whose *Miracula Arithmetica* reflects such a working-through. Descartes adopted a multiplicative conception as central, and developed the techniques for the needs of a subtle classification theory of constructional complexity of geometrical problems that went beyond what even Viète's students could do with algebraic methods.

Roth's key contribution and importance in the most influential line in the history of equations in the West is thus quite clear: Roth's *Arithmetica Philosophica* brought out for Descartes the multiplicative insights and degree uniformities on equations that he was to systematize and exploit. Viète's work seems not to have been an actual influence on Roth, Faulhaber, or Descartes's mastery of a multiplicative conception of equations.

3. Roth's seventh-degree problems

Section 3.1 translates Roth's six seventh-degree problems and comments on their solutions. (We omit the concluding mathematical coded-phrase puzzle [Wortrechnung].) Section 3.2 briefly draws together the discussion of the printed "errors."

The translated sections are consecutive in the original and include all comments Roth gave on the problems. The translation is literal; it replaces cossic notation for powers of the unknown quantity by exponent notation, but the terms are arranged as in the original so that the reader can reconstruct it. (For the first problem, this is illustrated with a reproduction from the original.) As before, items in square brackets are clarifications: an item in English supplements the German text; one in German gives the original of an English term.

The translation of each problem is followed by an analysis, in which we have attempted an overview of plausible solution methods, whether additively or multiplicatively motivated, that might have been intended by Roth, or might have been tried by Faulhaber and/or Descartes.[17] We try to indicate what difficulties each method would have to overcome, including the impact of the printing errors. The analyses use modern algebraic notation, but we try to indicate what might be done to carry out the solutions with the limited resources of cossic notation.

3.1. Translation and analysis

ERstlich/ 1 Bß + 584 zz + 17680 cc + 18416 z sein gleich 7 zcc + 266 ß + 158688 R + 174720. Wann nun der eine waare werth *radicis*, auß dieser æquation thut 10. So ist hierauff die Frag/ was/ welches/ vnd wie viel dann der andern/ noch waaren/ vnnd gedichten werth *radicis* seyn werden? Facit die andern seyn 10 + √48 vnd 10 ÷ √48 auch ÷ 4 vnd ÷ 7. sihest also/ daß diese æquation leidet drey waare vñ zween gedichte werth *radicis*.

Arithmetica Philosophica [190 recto]:

> *Further follow Questions on Seventh-Degree Equations [Bsurdesolida Coss].*
> I. First,
>
> $$1x^7 + 584x^4 + 17{,}680x^3 + 18{,}416x^2 \text{ are equal to } 7x^6 + 266x^5 + 158{,}688x + 174{,}720.$$
>
> Given, now, that the value of one true root of this equation is 10; then the question is what, which, and how many other values of the root, true and fictive, will there be? Answer [Facit]: the others are $10 + \sqrt{48}$, $10 - \sqrt{48}$, also -4 and -7. Thus you see that this equation admits three true and two fictive values of the root.

To get the roots, first set the equation equal to 0. Divide the polynomial by $x - 10$ to remove the given root. This leaves a sextic with constant term 17,472, i.e., $2^6 \cdot 3 \cdot 7 \cdot 13$. Trial-divide by each potential linear factor $x \pm n$ for divisors n of 17,472. Starting small, one succeeds with $x + 4$ and $x + 7$ (only), leaving a quartic polynomial:

$$624 + 384x - 176x^2 - 8x^3 + x^4.$$

[17] Some of the analysis, notably the exploration of problems with errors, utilized the computer program *Mathematica*; the formulas produced in this way are given in order of ascending powers of the unknown.

According to *Miracula Arithmetica* and the *Géométrie*, one would further substitute $x = z + 2$ to normalize the quartic:

$$640 - 384z - 200z^2 + z^4.$$

(If one did not normalize, the procedure would be similar.) Extracting its (real) roots gives the solution; but doing so will require the explicit quartic polynomial. Hence, one must actually divide by the two linear factors, not just discover and verify the integral roots $x = -4$ and $x = -7$.

Extract the remaining roots by an approach such as (a), (b), or (c):

(a) For the quartic problems from *Arithmetisch Cubiccossischer Lustgarten* (10 and 65), Roth used Ferrari's method following Cardano [1545, ch. 39]. He might have had the same in mind for this problem.

(b) Descartes' formulas for factoring quartic equations in the *Géométrie*, pp. 383–386, with signs properly disambiguated, gave an auxiliary bicubic:

$$-147{,}456 + 37{,}440y^2 - 400y^4 + y^6.$$

As we are looking to factor the quartic into quadratics with integer coefficients, we need a root $y^2 = n^2$ of the sextic, where n^2 divides $147{,}456 = 2^{14} \cdot 3^2$. One finds $y^2 = 16^2$, $y = \pm 16$, and the factorization $(52 - 20x + x^2)(x^2 + 12x + 12)$. The first factor gives $x = 10 \pm \sqrt{48}$; the second has no (real) roots. Supposing the factorization formulas to be available, there would be no problem using cossic notation because only one unknown occurs per polynomial.

(c) Before such formulas were at hand, Faulhaber [1622, chs. 42–44] and Descartes might have attacked the factorization of the (normalized) quartic into two quadratics by something like algebraic analysis with indeterminate coefficients (Section 2.4). By allowing a small finite search, one could, however, avoid both the elimination process and the auxiliary equation: a factorization would have the form

$$\bigl(z^2 - yz + v\bigr)\bigl(z^2 + yz + (640/v)\bigr),$$

where v divides $640 = 2^7 \cdot 5$. By symmetry, one can insist that (say) $v < 640/v$, leaving 16 possibilities for v (with sign). Pursuing each one separately, a cossist could use the numerical value instead of our unknown v. The (only) one that works out in the end is $v = 16$, which gives

$$\bigl(z^2 - yz + 16\bigr)\bigl(z^2 + yz + 40\bigr) = 640 - 24yz + \bigl(56 - y^2\bigr)z^2 + z^4 = 640 - 384z - 200z^2 + z^4,$$

and hence two separate conditions:

$$24y = 384; \qquad 56 - y^2 = -200.$$

Now $y = 16$ satisfies both conditions. After substitutions, this factors the original polynomial in x as indicated in (b).

The quartic in z and the pair of conditions on y involve only one unknown at a time. The formulas we have written with more than one unknown are only "needed" to give an overview of the relationships between the various cossically expressable polynomials in one unknown. A cossist would need to keep track of the rationale of the procedure without such aids. One often enough sees them do so successfully.

The original polynomial factors as

$$(x-10)(x-4)(x-7)\left(x^2-20x+52\right)\left(x^2+12x+12\right).$$

> II. Just so, the true *myriagonal* root [wurtzel] out of
>
> $$1x^7+904x^4+3990x^3+24{,}162x+100{,}800-12x^6-79x^5-33{,}165x^2$$
>
> is 3. But because the cossic quantities now have more than one *myriagonal* root [wurtzel], I ask accordingly which and how many there will be of those. Answer: these cossic quantities admit 5 *myriagonal* roots, namely four true ones, which are $\sqrt{39}+6$, and just so 8, 5, and 3, and then a fictive one, which is $5-\sqrt{39}$.

The printed polynomial does not have 3 as a root, nor is the printed pair of irrational roots properly conjugate. They should be either $5\pm\sqrt{39}$ or $6\pm\sqrt{39}$, and in either case one is positive and the other negative.

Using the indicated solution and separately trying out 5 (shown below) and 6 for the rational part of the binomial, the problem can be restored. Multiplying the polynomials for the given roots,

$$\left(x-(5+\sqrt{39})\right)\left(x-(5-\sqrt{39})\right)(x-8)(x-5)(x-3),$$

and assuming that the constant term (root-product) and the x^6-term (root-sum) are correct as printed, we find for the co-factor $x^2+14x+60$. Multiplying these factors, one obtains

$$100{,}800+29{,}160x-38{,}164x^2+3990x^3+904x^4-79x^5-12x^6+x^7.$$

This polynomial agrees with the printed one except for two digits each in two large-coefficient terms, and does have 3 as a root. In contrast, $6\pm\sqrt{39}$ for the irrational roots leads to a polynomial with quite different coefficients. The proper irrational root should thus be $\sqrt{39}+5$.

With the corrected polynomial, the solution goes as in Problem I. Once the linear factors have been extracted, the quartic quotient has constant term $840=2^3\cdot 3\cdot 5\cdot 7$. As before, symmetry leaves 32 possibilities (with sign) to be tried for the constant term of one quadratic factor. The corrected polynomial factors as

$$(x-3)(x-5)(x-8)\left(x^2-10x-14\right)\left(x^2+14x+60\right).$$

> III. Just so, there are some cossic quantities, as
>
> $$1x^7+9264x^3+13{,}617x^2-3x^6-160x^5-50x^4-159{,}168x-201{,}600.$$
>
> Now the question is, how these are to be changed (but so that they with their numbers [coefficients] and cossic signs [*characteribus*] remain unchanged, rather [they are changed] only by the signs + and −) [190 verso] so that from the quantities given above, the true *hecatogonal* roots (each for itself) [will] be equal to the fictive roots [wurtzeln] of the quantities into which they are changed. And conversely [herwiderumben], that in this [changed] one, the true ones are equal to the fictive roots in the equation displayed above.

> Answer: [the modified roots come] out of
>
> $$1x^7 + 9264x^3 + 3x^6 + 50x^4 + 201{,}600 - 13{,}519x^2 - 160x^5 - 159{,}168x.$$
>
> The true *hecatogonal* roots [of this equation] are the fictive ones of the equation given [first] above, and the true ones in the one above are the fictive roots in this [equation].

If the coefficients of the printed polynomials had agreed up to signs in the right way, one could just read off a correct answer—a short-cut solution. The discrepancy between the quadratic terms (13,617 vs 13,519) indicates that at least one printed polynomial contains an error. (Trying as root values the factors of the constant term, one also finds that neither polynomial has an integer root.)

The simplest restoration notices that the constant term is highly composite (suggesting the polynomial can be factored with integer coefficients); indeed, except for the opposite sign, it is plainly double the constant term from the previous problem. This suggests that the polynomial might arise by doubling one of the roots in Problem II and changing its sign. Indeed, changing the root 3 in II to -6 produces just the right change in the sum of the roots (from 12 to 3). As 3 was the originally given root in II, the reader should already have the sextic that results there by dividing by $x - 3$; multiplying it by $x + 6$ gives

$$(-8 + x)(-5 + x)(6 + x)\left(-14 - 10x + x^2\right)\left(60 + 14x + x^2\right),$$
$$-201{,}600 - 159{,}120x + 13{,}568x^2 + 9264x^3 - 50x^4 - 160x^5 - 3x^6 + x^7,$$

a compelling reconstruction of the first printed polynomial.

Given the factorization, one can carry out appropriate sign transformations according to the established principles for linear and quadratic equations. Remultiplying the modified factors gives the signs as in the second printed polynomial. But now one has a procedure and rationale.

Might one instead guess that the correct quadratic coefficient is the average of the two printed ones? This leaves one, at best, alerted by the then repeated low digits (68) that the coefficient of the linear term may also need correction. Might that then be accomplished painfully by trial and error or a few very good guesses? It would still be awkward to recognize the correct linear term: if one did not have it right, how would one tell? The polynomial would not factor well; only by trying new values for the linear term and seeing whether the factorization was better could one decide to stop. What if one did not recognize that the polynomial "should" factor into integral linear and quadratic factors?

An elaborate algebraic analysis would also eventually lead to the desired polynomial. Neither of these last proposals seems plausible, in light of the one above.

> IV. Just so, there are some cossic quantities which have seven *tessaracosiagonal* roots [wurzeln]; namely, five true ones, they are $3 + \sqrt{3}$, $3 - \sqrt{3}$, 10, 5, 3. Then also there are two fictive [ones], as -6 and -11. Now the question is, what were the quantities from which these roots were to be taken? Answer:
>
> $$1x^7 - 7x^6 - 133x^5 + 1141x^4 + 1188x^3 - 30{,}359x^2 + 81{,}522x - 59{,}400.$$
>
> Here there might be many who would say, how could it be possible, if only the values of the roots are known to one, that from this one finds their quantities or equation? Those, however, should know that this is not the fault of the Art [Kunst], but rather of their own ignorance. For there is indeed a pretty [schöne] rule through which (as long as one is aware of all values of the root in an equation) this desire can be

> accomplished [vollführt]. You may figure it out yourself: just as it is possible to find the values of the root from an equation, so it is conversely possible to seek an equation from all the values of its roots.

Multiplication gives

$$\big(x-(3+\sqrt{3})\big)\big(x-(3-\sqrt{3})\big)(x-10)(x-5)(x-3)(x+6)(x+11)$$
$$=-59{,}400+81{,}720x-30{,}558x^2+1188x^3+1141x^4-133x^5-7x^6+x^7.$$

Four digits (in two large-coefficient terms) are incorrect in Roth's printed answer, just to make sure one has to do the multiplication.

> V.[18] Just so,
>
> $1x^7+19x^6+94x^5$ are equal to $82x^4+1655x^3+4861x^2+5208x+$ some unknown lone numbers.
>
> The true value of its root is a *binomium* [i.e., of the form $M+\sqrt{N}$]; of which value of the root the rational [rational] part, namely 2 (but nothing more), is known. The question about this is now, how much is the Surdic part of the indicated value of the root, and what had the unknown lone numbers [constant term] in the equation been? Answer: the Surdic part of the value of the root is $\sqrt{5}$, and the lone numbers in the equation are 980.

To solve the problem by extending Roth's additively motivated method (footnote 18), we substitute $2+x$, where x is the irrational part of the root sought, into the equation. Abbreviating "some unknown lone numbers" by L, this gives

$$(2+x)^7+19(2+x)^6+94(2+x)^5=82(2+x)^4+1655(2+x)^3+4861(2+x)^2+5208(2+x)+L,$$

which after much arithmetic evaluates to

$$-40{,}060-L-35{,}520x-4007x^2+5049x^3+2278x^4+406x^5+33x^6+x^7=0.$$

As x is a square root of a rational number, its even powers are rational and its odd powers surds; each of these two sums must separately equal 0. Dividing all odd powers by x and substituting z for x^2, we get

$$-35{,}520+5049z+406z^2+z^3=0.$$

By trial division on the small factors of 35,520, or other cubic methods, one finds a rational root $z=5$. Hence $x=\sqrt{5}$. Substituting this into the sum of the even powers gives $L=980$.

[18] Faulhaber remarked in *Miracula Arithmetica*, p. 59, that this one was a bit harder than the previous one, but solvable "in an orderly and systematic fashion" by a special display/calculation [ein sonderbare Tafel] analogous to that for his [1604] cubic problems 66–67. These had been given additively motivated solutions by Roth [1608, Part 2, fol. 81.verso–84.recto]; but inmediately continuing in chs. 42–44 discussed above, *Miracula Arithmetica* gave a new, multiplicative treatment of those problems. We infer that Faulhaber and Descartes solved Problems V and VI by the multiplicative approach indicated below, rather than the additive one inspired by Roth's solutions to the earlier problems that we indicate first.

Cossically, one reuses the same symbols for the powers of x and z and handles the odd and even powers separately without giving either of the displayed equations whole.

From a multiplicative point of view, the problem poses similar conditions. We cannot start by extracting linear factors, because we do not know the constant term. But given that one solution is of the form $2+\sqrt{n}$ for some integer n, the corresponding quadratic factor must be of the form $x^2-4x+(4-n)$. Divide the septic by this quadratic factor; the remainder is

$$\bigl(-L+30{,}980-14{,}105n+1466n^2+31n^3\bigr)+\bigl(-35{,}520+5049n+406n^2+n^3\bigr)x.$$

This must vanish if the quadratic is a factor of the septic. As before, find the root $n=5$ of the coefficient $-35{,}520+5049n+406n^2+n^3$ of x: it factors as $(-5+n)(7104+411n+n^2)$, and hence vanishes for $n=5$ and no other integer. Thus, for the remainder to vanish the quadratic factor must be $x^2-4x+(4-5)=x^2-4x-1$; $30{,}980-14{,}105n+1466n^2+31n^3$ then evaluates to 980. To cancel this, the original constant term L must be 980 and the binomial root must be $2+\sqrt{5}$. Its conjugate $2-\sqrt{5}$ is indeed negative. The polynomial now factors as

$$(2+x)(7+x)(10+x)\bigl(-1-4x+x^2\bigr)\bigl(7+4x+x^2\bigr).$$

Just as in the additive version, a cossic presentation might avoid displaying the equation in which x and n appear at the same time, and reuse the cossic unknown for both.

> VI. Just so again,
>
> $$1x^7+15x^6+16x^5+8538x+1040[\text{lone numbers}]+\text{some squares, is equal to } 574x^4+1847x^3.$$
>
> The value of the root is a *binomium* $[M+\sqrt{N}]$ or *residuum* $[M-\sqrt{N}]$, of which no more is known than the Surdic part, namely $\sqrt{3}$. Accordingly, the question is, how much was the rational part of the value of the root indicated, and what [number] stood in the equation for squares? Answer: the rational part of the value of the root was 4, and in the equation stood $2531x^2$.

Following the same additive tack as in Problem V, one would evaluate the result of substituting $x+\sqrt{3}$ in the given polynomial:

$$\begin{aligned}&1040+8538(x+\sqrt{3})+\text{some }(x+\sqrt{3})^2+16(x+\sqrt{3})^5+15(x+\sqrt{3})^6+(x+\sqrt{3})^7\\&\quad-1847(x+\sqrt{3})^3-574(x+\sqrt{3})^4\end{aligned}$$

equals

$$\begin{aligned}&-3721+3168\sqrt{3}-7176x-6078\sqrt{3}x-8307x^2-4872\sqrt{3}x^2-1052x^3-1396\sqrt{3}x^3+101x^4\\&\quad+185\sqrt{3}x^4+79x^5+90\sqrt{3}x^5+15x^6+7\sqrt{3}x^6+x^7+3\text{ times some}+2\sqrt{3}x\text{ times some}+\text{some }x^2\end{aligned}$$

with the understanding that "some" is the same positive integer in all three places. For this to be zero, both the rational and the irrational part must separately be zero:

$$-3721-7176x-8307x^2-1052x^3+101x^4+79x^5+15x^6+x^7+3\text{ times some}+\text{some }x^2=0,$$

$$3168 + 2(-3039 + \text{some})x - 4872x^2 - 1396x^3 + 185x^4 + 90x^5 + 7x^6 = 0.$$

From this point on, the solution might proceed as in (a) or (b):

(a) Using only that integer roots must evenly divide the constant term, $3168 = 2^5 \cdot 3^2 \cdot 11$ for the second equation; the constant term of the first equation remains unknown. Trying $x = n$ for each divisor n of 3168 (starting small), solve the second equation for "the number," and check whether this and the same divisor n make the first polynomial zero. This works for $x = 4$ and $2531x^2$ (and for no other divisor).

(b) Allowing elimination of the (linearly occurring) "some" between the equations,

$$9504 - 10{,}792x + 2904x^2 + 6348x^3 - 2213x^4 - 1328x^5 + 48x^6 + 60x^7 + 5x^8 = 0,$$

where $9504 = 3 \cdot 3168$. For each divisor n of 3168, starting with the small ones, evaluate the polynomial to find the root 4. (There are no other integral roots.) Substituting this into either polynomial, we find 2531 for the number of squares.

The root could be a residuum instead, but repeating for $x - \sqrt{3}$ gives no further solution.

There is also, however, a multiplicatively motivated approach to Problem VI, via the quadratic factor resulting from a conjugate pair of irrational roots. One checks that the only (integral) factors n of the constant term 1040 for which $n + 3$ is a square are 1 ($n + 3 = 4$) and 13 ($n + 3 = 16$). These give potential quadratic factors $x^2 - 2\sqrt{n+3}\,x + n$, namely $x^2 - 4x + 1$, respectively $x^2 - 8x + 13$. After division by either of these potential factors, the remainder gives two linear conditions on the unknown quadratic coefficient "some" that must be simultaneously satisfied for it to vanish. For $n = 1$, these are incompatible ("some"$= 12{,}227$ and "some"$= 8339$); for $n = 13$, both conditions give "some"$= 2531$. This gives the quadratic term and the required root.

The polynomial now factors as $(2 + x)(5 + x)(8 + x)(13 - 8x + x^2)(1 + 8x + x^2)$.

3.2. *Errors in the printed problems*

Problems II, III, and IV were printed with errors. Upon correction, the errors initially appear to be typos or printer's misreadings of one or two digits in large-coefficient terms. But upon review of all the "errors" in the set, their pattern is too improbable to dismiss as poor transcription.

The printed polynomial in II was too small by $4998x$ and $4999x^2$, the first printed polynomial in III was larger than the second by $98x^2$ and both were too large by $48x$, the printed polynomial in IV was too small by $198x$ and $199x^2$. The corrected quadratic term in III is the average of the two printed quadratic terms, so *both* were off by $49x^2$; its last two digits, 68, may also be found in the (wrong) printed linear term, 159,168. All this is no mere accident of the printer's art.

The differences cannot be accounted for by any systematic misreading of particular digits in the author's handwriting: there is no pattern of substitutions (say, "1" for "6").

Problems II and III cannot be solved until the errors are corrected; one recognizes at the outset that there is an error. With the right multiplicative technique, Problem IV may be solved directly, after which the error in the printed answer is evident.

We conclude that the errors were by Roth's careful design and the restorations must have been intended as part of the problems. This made Problem II challenging beyond Problem I even though they initially appear to be of the same type. Similarly, the errors in III forced one to apply algebraic insights well beyond the sign-changing technique; and in the process of finding the correct polynomial one came to have its factors, so that the solution technique could be verified and understood. In IV, the printed answer gave a polynomial in which a few wrong digits served to prevent claims of success by a bogus construction of the printed polynomial; meanwhile, the correct answer, once found, is so close to the printed one as to confirm the solution.

References

Adam, Charles, Tannery, Paul, 1996. Oeuvres de Descartes, xi vols. Vrin, Paris. Cited from this reprint edition, as (e.g.): AT X 50 (vol. X, p. 50).

Baillet, Adrien, 1691. La vie de Monsieur Des-Cartes. Paris. 2 vols.

Bos, Henk, 1990. The Structure of Descartes's *Géométrie*. In: Giulia, Belgioso, et al. (Eds.), Descartes: il metodo e i saggi.... Armando Paoletti, Florence, pp. 349–369. As reprinted in Bos [1993], pp. 37–57.

Bos, Henk, 1993. Lectures in the History of Mathematics. American Mathematical Society, Providence, RI.

Bos, Henk, 2001. Redefining Geometrical Exactness. Descartes' Transformation.... Springer-Verlag, Berlin/New York.

Cajori, Florian, 1928. A History of Mathematical Notations. Open Court, Chicago. Reprinted by Dover, New York, 1993.

Cantor, Moritz, 1880–1908. Vorlesungen uber Geschichte der Mathematik. Teubner, Leipzig. Quoted from the second ed., vol. 2, 1900.

Cardano, Girolamo, 1545. Ars Magna Nuremberg. Roth, 1608 (cf. fol. 4r.) refers to the augmented second ed., Basel, 1570.

Clavius, Christopher, 1608. Algebra. Rome.

Descartes, René, 1618–1619. Correspondence with Beeckman. AT 154–164.

Descartes, René, 1637. La *Géométrie*. AT VI. In facsimile and translate, in Latham and Smith [1925].

Faulhaber, Johann, 1604. Arithmetisch Cubiccossischer Lustgarten. Tübingen. For the full title, see the title page reproduced in Schneider [1993, p. 55] and Hawlitschek [1995, p. 109].

Faulhaber, Johann, 1620. Continuatio des Newen Mathematischen Kunstspiegels. Dieterich Werlin, Tübingen.

Faulhaber, Johann, 1622. MIRACULA ARITHMETICA zu der Continuation seines Arithmetischen Wegweisers gehörig. Zu Augspurg, bey David Francken. ANNO CHRI. M. DC. XXII. Title page reproduced in Hawlitschek [1995, p. 66].

Faulhaber, Johann, 1631. *Academia Algebrae*. Darinnen die miraculossische Inventiones zu den höchsten Cossen weiters continuirt.... Augsburg. Title page reproduced in Hawlitschek [1995, p. 178].

Hawlitschek, Kurt, 1995. Johann Faulhaber 1580–1635. Eine Blütezeit der mathematischen Wissenschaften in Ulm, Band 18. Schriftenreihe der Stadtbibliothek Ulm. Includes selected correspondence of Faulhaber, notably with Sebastian Kurz in Nuremburg [BN Paris, located by Schneider].

Harriot, Thomas, 1631. Ars analyticae praxis. London.

Huygens, Christiaan, 1888–1950. *Oeuvres complètes...*, Nijhoff, La Haye. Cited as [HO].

Latham, Martha, Smith David, Eugene, 1925. The Geometry of René Descartes. Open Court. Reprinted by Dover, New York, 1955.

Lipstorp, Daniel, 1653. Specimina philosophiae cartesianae. Leyden. The fragment on Descartes and Faulhaber, pp 78–80, is transcribed in AT X 252–253.

Mahoney, Michael, 1980. The beginnings of algebraic thought in the seventeenth century. In: Gaukroger (Ed.), Descartes: Philosophy, Mathematics and Physics. Harvester/Barnes & Noble, Brighton/Totowa, NJ, pp. 141–155.

Manders, Kenneth, 1995. Descartes et Faulhaber. In Bulletin Cartésien XXIII. Archives de Philosophie 58 (3), 1–12. (English translation available.)

Roth, Peter, 1608. Arithmetica Philosophica. Nuremberg. For the full title, see the title page reproduced in [Schneider, 1993, p. 87] .

Rudolff, Christoff, 1525. Behend und hübsch Rechnung durch die kunstreichen Regeln Algebrae, so gemeincklich die Coss genennt werden. Strassburg. Expanded and revised edition published by Stifel, Königsberg 1553, with subsequent editions to 1615.

Petri, Nicolaus, 1583–1663. Practique on te leeren reekenen Amsterdam. *Miracula Arithmetica* refers to the 1598 edition.

Sasaki, Chikara, 2003. Descartes's Mathematical Thought. Kluwer, Dordrecht.

Schneider, Ivo, 1993. Johann Faulhaber 1580–1635. Birkhäuser, Basel. Includes a survey of sources on Faulhaber.

Smith, David Eugene, 1929. A Source Book in Mathematics. McGraw–Hill, New York. Reprinted by Dover, New York, 1959, in 2 vols.
Stifel, Michael, 1543. Arithmetica Integra. Nuremberg. Five subsequent editions, to 1586.
Struik, Dirk, 1969. A Source Book in Mathematics. Harvard Univ. Press, Cambridge, MA.
Treutlein, Peter, 1877. Die deutsche Coss. Zeitschr. Math. Physik 22 (Supplement).
Tropfke, Johannes, 1902–1903. Geschichte der Elementar-mathematik in systematischer Darstellung. Veit, Leipzig. Second ed., vol. 3, Walter de Gruyter, Berlin/Leipzig, 1922.
van Maanen, Jan, 1987. Facets of seventeenth century mathematics in the Netherlands. Ph.D. dissertation, Utrecht.
Viète, François, 1615. De Aequationum recognitionum et emendatione, edited by A. Anderson. Paris.
Viète, François, 1983. The Analytic Art, translated by T. Richard Witmer. Kent Sate Univ. Press, Kent, OH.